A Deterministic Algorithm for Counting Colorings with 2Δ Colors

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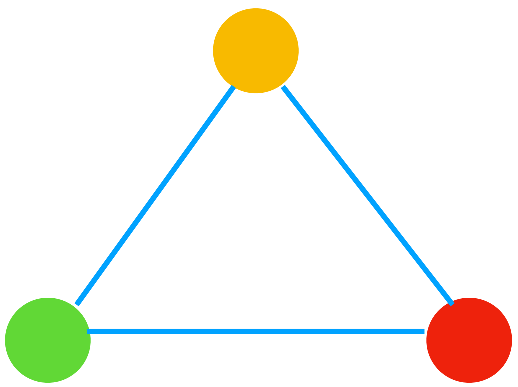
Approximate counting

- Consider proper q-colorings of a graph G
- Let Δ:=Δ(G) be the maximum degree
- If $q \geq \Delta + 1$:
  - G has a q-coloring, which can be found efficiently
  - How many q-colorings does G have?



#P-complete, approximation?

Approximate counting

- Compute $(1 \pm \varepsilon) \cdot (\# q\text{−colorings})$
- Equivalent to
  - **(Approximate) sampling**

    Sample a q-coloring uniformly at random?

  - **Approximate inference**

    Given the color of a vertex u,

    what can you infer about the color of another vertex v?

  - **Approximate root-finding**
  - …

Our main result

<u>Our algorithmic result:</u>
For any integer $q \geq 2\Delta$, and bounded-degree graph G, there exists a *deterministic FPTAS*, which outputs $\hat{Z}$ s.t.

$$\hat{Z} \in (1 \pm \varepsilon)(\#q\text{−colorings}(G))$$

in time $\mathrm{poly}(|G|, 1/\epsilon)$ for G

<u>Triangle-free graphs:</u>
For any integer $q \gtrapprox 1.7633\Delta + \beta$, and triangle-free graph G of bounded degree, the above FPTAS also works

Dobrushin condition

[Gamarnik, Katz and Mishra] 1.7633 is closely related to *<u>strong spatial mixing</u>*, which is the unique positive solution of the equation

$$xe^{-1/x} = 1$$

<u>Prior works (incomplete list):</u>

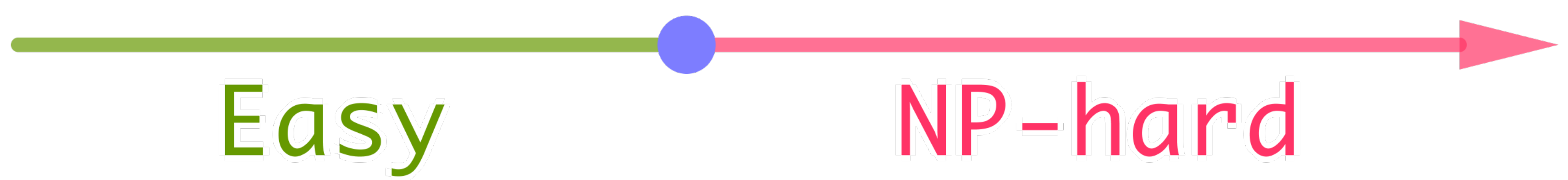
- ***Randomized*** **MCMC for $q \geq 2\Delta + 1$ [Jerrum]**
- ***Randomized*** **MCMC for $q \geq (\frac{11}{6} - \epsilon)\Delta$ [Chen et al.] [Vigoda]**
- **Deterministic algorithm for $q \geq 2.58\Delta + 1$ [Lu and Yin]**
- **Deterministic algorithm for $q = 4, \Delta = 3$ [Lu et al.]**

Computational phase transition

Our work: exploit a formal connection between algorithms and phase transition.

Crucially we exploit both directions

- Many problems have a "computational phase transition" (roughly):

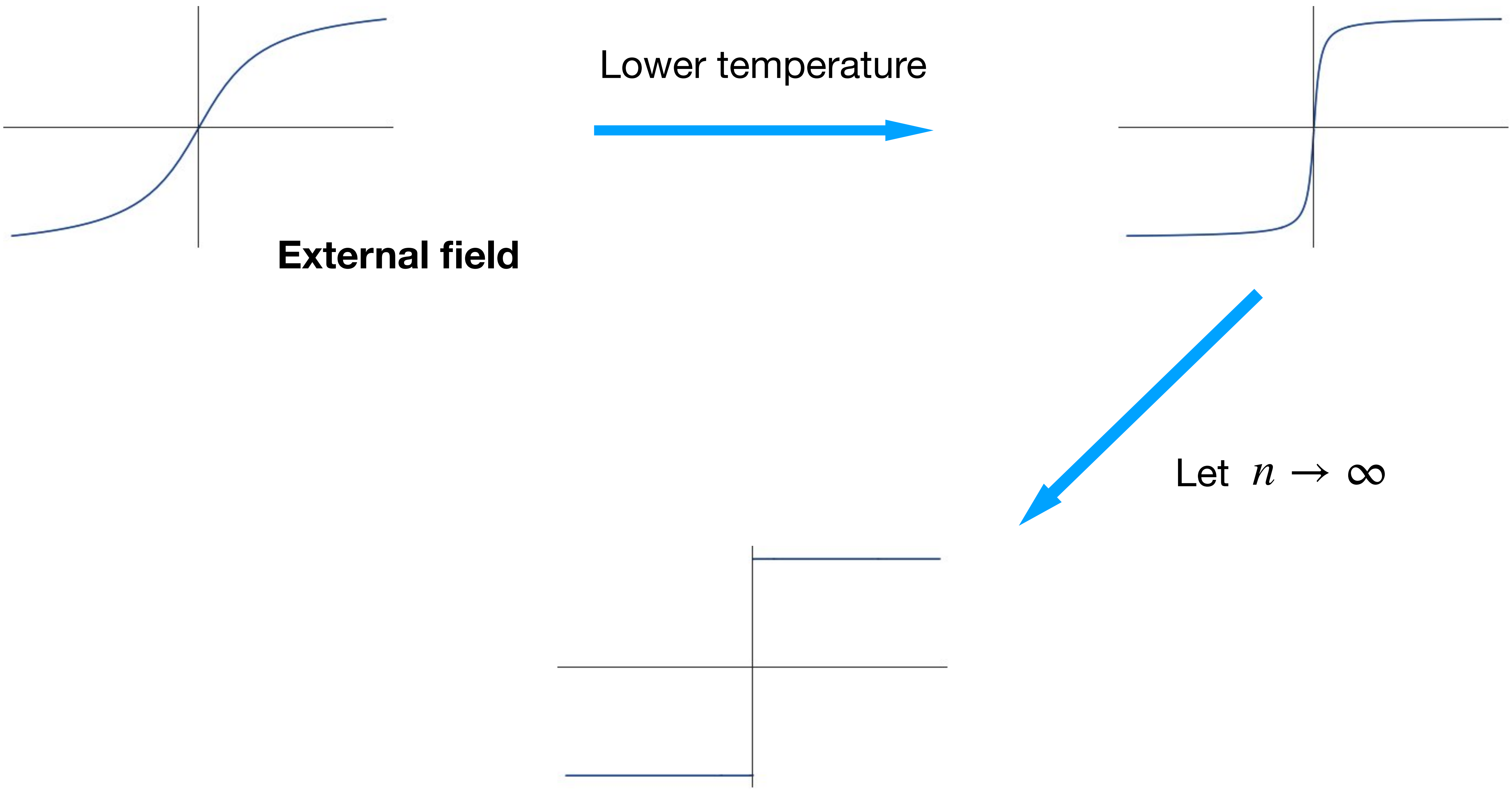


- In some cases, known to coincide with statistical physics phase transition:
  - Hardcore model
  - Anti-ferromagnetic Ising/2-spin model
  - ....

Phase transitions ≈ Discontinuity

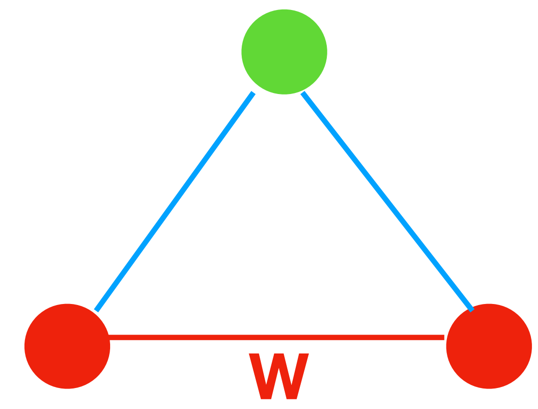
In statistical physics

Observable, e.g., mean magnetization :≈ average number of +-spin vertices



Approximate counting

The Potts model



- A graph G=(V,E)
- Configuration $\sigma : V \to [q]$
- The partition function

$$Z_G(w) := \sum_{\sigma:V\to[q]} w^{(\#\text{monochromatic edges})}$$
$$= \sum_{\sigma:V\to[q]} w^{|E_=(\sigma)|}.$$

Example:

- $Z_G(0) = (\#\text{ valid } q\text{-colorings of } G)$
- **q=2: Ising model**

- Gibbs distribution:

$$\Pr[\sigma] = \frac{1}{Z_G(w)} \cdot w^{|E_=(\sigma)|}$$

Goal: compute $(1 \pm \varepsilon) \cdot Z_G(w)$ **.**

Phase transitions formally:

Geometry of polynomials

- Lee-Yang theory:

Phase transition ≈ complex zeros of Z

- Recall the partition function: $$Z_G(w) := \sum_{\sigma:V\to[q]} w^{|E_=(\sigma)|}$$

- Observable:

$$\mathbb{E}_\sigma|E_=(\sigma)| = \sum_{\sigma:V\to[q]} |E_=(\sigma)| \cdot \Pr[\sigma]$$
$$= w \cdot \frac{\partial \log Z_G(w)}{\partial w}$$

- Analyticity of log Z ≈ Continuity of observables

Lack of phase transition ≈ Lack of complex zeros

Algorithms from Phase transitions: Barvinok's interpolation

Recall the partition function: $Z_G(w) := \sum_{\sigma:V\to[q]} w^{|E_=(\sigma)|}$

[Barvinok, Barvinok and Soberon]

- Consider the Taylor expansion of log Z
- In a zero free region, log Z can be approximated to $\pm\epsilon$ by its k-th order Taylor series for $k = O(\log(n/\epsilon))$
- $\log Z_G(w) \pm \epsilon \iff (1 \pm \epsilon) \cdot Z_G(w)$
- k-th order Taylor series is determined by the first k+1 coefficients of Z
- Naively computing the first k+1 coefficients of Z takes time $O(n^k)$
- $\implies$ Quasi-polynomial time algorithm for $k = O(\log(n/\epsilon))$
- Exploiting the combinatorial structure speeds up to $O(n(e\Delta)^k)$

Fisher zeros of the Potts model

Recall the partition function:

$$Z_G(w) := \sum_{\sigma: V \to [q]} w^{|E_=(\sigma)|}$$

Our zero-freeness result:

Fix any integer q such that $q \geq 2\Delta$. Then there exists a constant τ_Δ such that $Z_G(w) \neq 0 \text{ when } w \in B([0,1], \tau_\Delta)$

Prior to our work:

- $q \geq e\Delta + 1$ **[Bencs. et. al.]: similar region**
- $q \geq 7.964\Delta$ **[Sokal] or** $q \geq 6.907\Delta$ **[Fernández and Procacci]: entire unit disk**

Potts model (triangle-free)

$$Z_G(w) := \sum_{\sigma: V \to [q]} w^{|E_=(\sigma)|}$$

Triangle-free graphs:

Fix any integer q such that: $q \gtrsim 1.7633\Delta + \beta$, there exists a constant τ_Δ such that

$$Z_G(w) \neq 0 \text{ if } G \text{ is triangle-free and } w \in B\left([0,1], \tau_\Delta\right)$$

$\Longrightarrow$ **FPTAS for bounded degree graphs**

Phase transitions from analysis of algorithms

Zero-freeness using algorithmic ideas

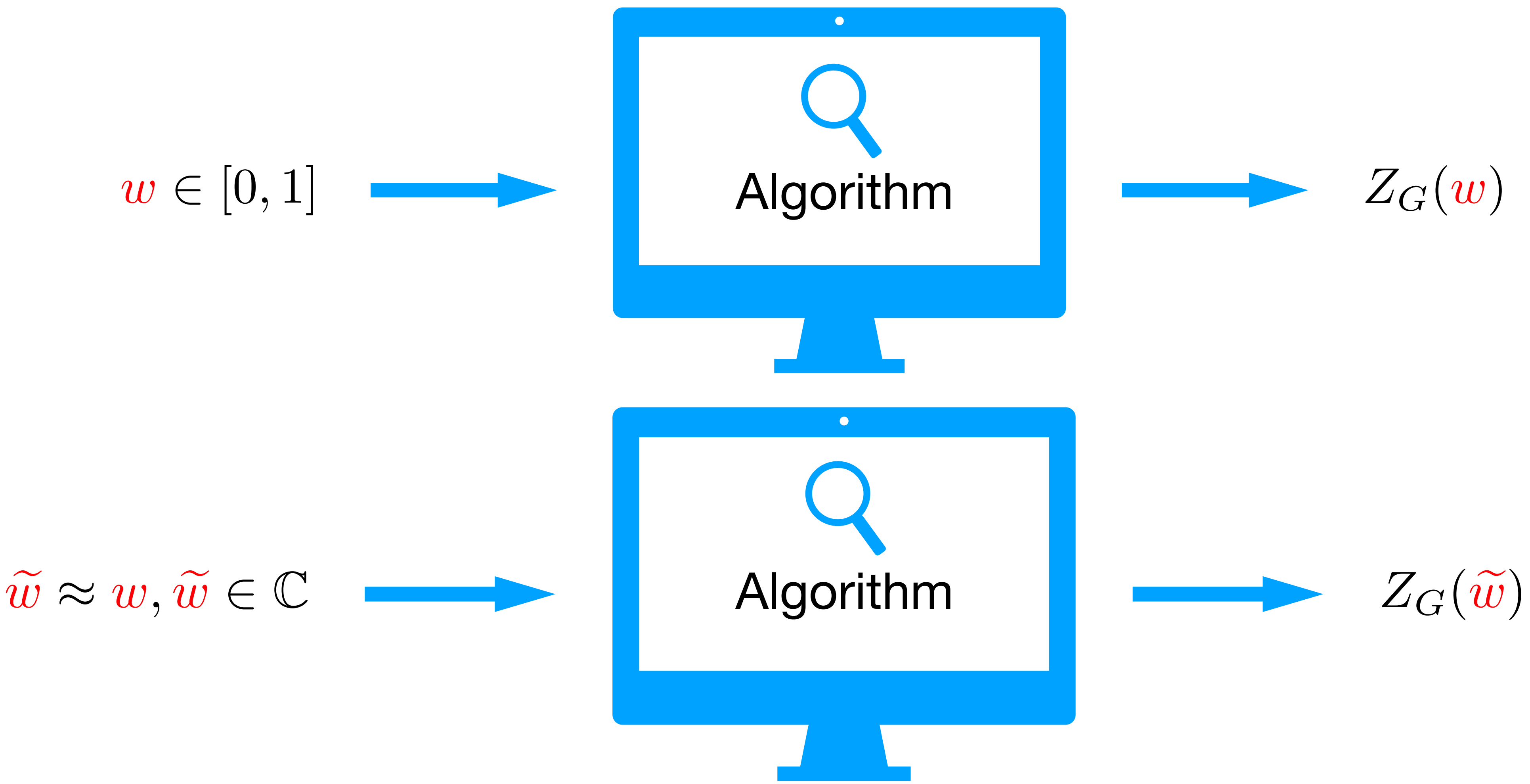
$$Z_G(w) := \sum_{\sigma : V \to [q]} w^{|E_=(\sigma)|}$$

If G has a valid q-coloring, then

$$Z_G(w) \geq 1$$

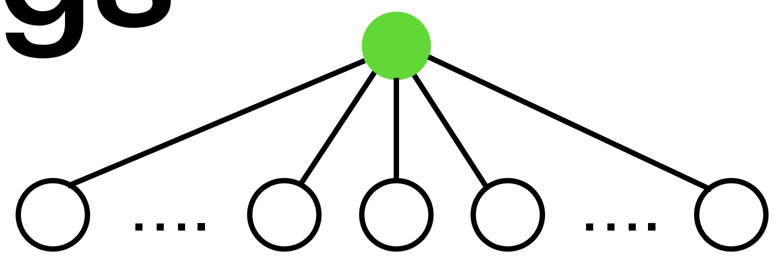
It suffices to prove that

$$\left| \frac{Z_G(\widetilde{w})}{Z_G(w)} \right| > 0$$



Ratios and Pinnings

Fix a vertex v



$Z_{G,v}^{(i)}(\omega)$: colorings in which vertex v receives color **i**

$$Z_G(\omega) = \sum_{i \in L(v)} Z_{G,v}^{(i)}(\omega)$$

$$= Z_{G,v}^{(j)}(\omega) \cdot \sum_{i \in L(v)} \frac{Z_{G,v}^{(i)}(\omega)}{Z_{G,v}^{(j)}(\omega)}$$

$$Z_G(\widetilde{\omega}) = \sum_{i \in L(v)} Z_{G,v}^{(i)}(\widetilde{\omega})$$

$$= Z_{G,v}^{(j)}(\widetilde{\omega}) \cdot \sum_{i \in L(v)} \frac{Z_{G,v}^{(i)}(\widetilde{\omega})}{Z_{G,v}^{(j)}(\widetilde{\omega})}$$

Induction

Consider $R_{G,v}^{(i,j)}(\omega) := \dfrac{Z_{G,v}^{(i)}(\omega)}{Z_{G,v}^{(j)}(\omega)}$ **inductively**

We show that $R_{G,v}^{(i,j)}(\widetilde{w}) \approx R_{G,v}^{(i,j)}(w)$

Tree recurrence

$w \in [0,1]$ →  → $R_{G,v}^{(i,j)}(w)$

At every vertex of the tree, the two computations remain close

$\widetilde{w} \approx w, \widetilde{w} \in \mathbb{C}$ →  → $R_{G,v}^{(i,j)}(\widetilde{w})$

We treat the tree recurrence as a complex dynamical system:

- Not every orbit is well-behaved (unless $q \gg 2\Delta$): main barrier
- Insight: There exists a "***high entropy***" (nice) orbit, which is well-behaved

High entropy condition

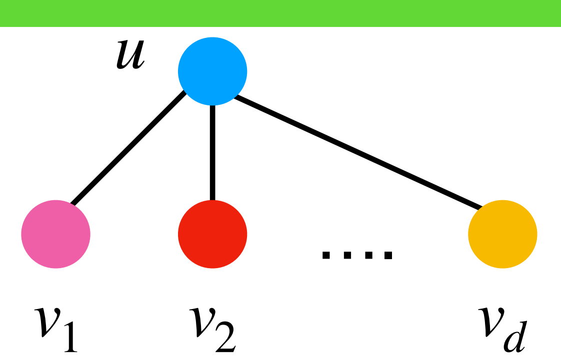
Consider list-coloring

For any vertex u in G, any $w \in [0,1]$ and any color i,

$$\Pr_{G,w}[c(u) = i] \le \frac{1}{\Delta + 2}$$

Lemma:
For any list-coloring instance such that $q \ge 2\Delta + 2$, or the graph is triangle-free and $q > 1.76\Delta + \beta$, then our "high entropy condition" holds

A more careful argument shows that $q \ge 2\Delta$ suffices



Conjecture:
For graphs of higher girth, $q \ge \alpha\Delta + \beta$ for a smaller α should suffice

Summary of the induction

$$Z_G(w) = \sum_{i \in L(v)} Z_{G,v}^{(i)}(w)$$

$$= Z_{G,v}^{(j)}(w) \cdot \sum_{i \in L(v)} \frac{Z_{G,v}^{(i)}(w)}{Z_{G,v}^{(j)}(w)}$$

$$Z_G(\widetilde{w}) = \sum_{i \in L(v)} Z_{G,v}^{(i)}(\widetilde{w})$$

$$= Z_{G,v}^{(j)}(\widetilde{w}) \cdot \sum_{i \in L(v)} \frac{Z_{G,v}^{(i)}(\widetilde{w})}{Z_{G,v}^{(j)}(\widetilde{w})}$$

Inductively

$$\left| \frac{Z_{G,v}^{(j)}(\widetilde{w})}{Z_{G,v}^{(j)}(w)} \right| \geq e^{-\epsilon(n-1)}$$

$$R_{G,v}^{(i,j)}(\widetilde{w}) \approx e^{\epsilon} \cdot R_{G,v}^{(i,j)}(w)$$

Recap

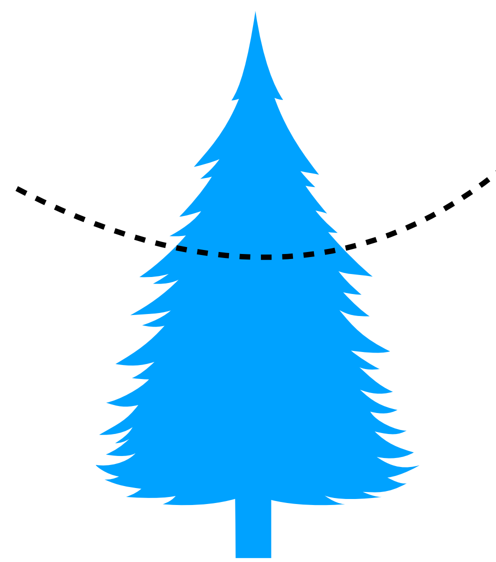
- One can exploit the *analysis of algorithms* (**tree recurrence**) to study *phase transitions* (prove zero-freeness)

$$\frac{Z_{G,v}^{(i)}(w)}{Z_{G,v}^{(j)}(w)} = \frac{Z_{G,v}^{(i)}(\widetilde{w})}{Z_{G,v}^{(j)}(\widetilde{w})}$$

- Conversely, zero-freeness results can also be exploited algorithmically (via Barvinok's interpolation)

Discussion: Comparison to decay of correlations

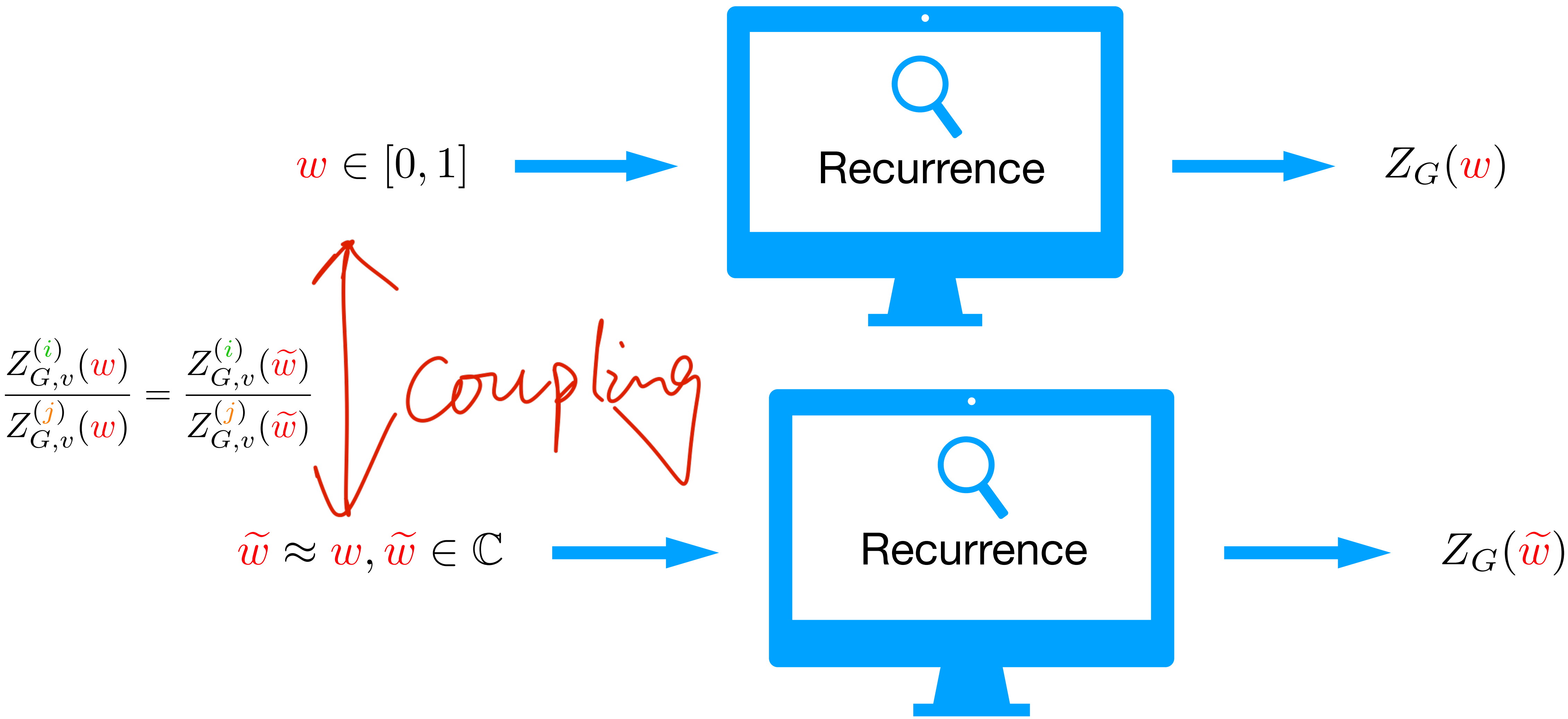
Correlation decay approach



Issue: guessing the values explicitly can be hard

In proving zero-freeness, we only need that "*good*" values exist

Comparison to MCMC



Future direction:

- Generalize to more sophisticated coupling argument
- De-randomize MCMC

Thanks

Q & A