Fisher zeros and correlation decay in the Ising model

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Motivation

In the past few years:
Phase transitions in statistical physics → algorithms

In this work, we study the converse:
Can we study phase transitions in statistical physics via algorithmic techniques?

Ising model

- Configuration: $\sigma \in \{+,-\}^V$
- Edge potentials: $\varphi_e(\sigma_u, \sigma_v) = \begin{cases} \beta & \text{if } \sigma_u \neq \sigma_v \\ 1 & \text{otherwise} \end{cases}$

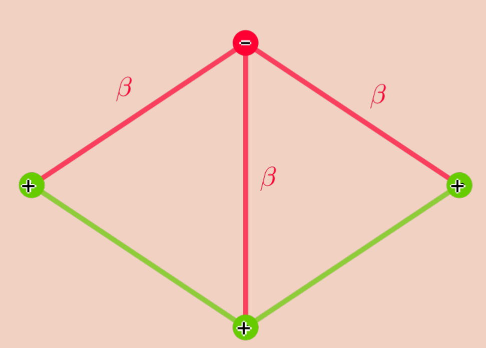

Fig. 2: A spin configuration with weight β^3

Ising model as cut generating polynomial

$$Z_G(\beta) = \sum_{S \subseteq V} \beta^{|E(S, V \setminus S)|} = \sum_{k=0}^{|E|} \gamma_k \beta^k$$

where γ_k := number of k-edge cuts

Gibbs distribution: $\Pr[(S, V \setminus S)] = \frac{1}{Z_G(\beta)} \cdot \beta^{|E(S, V \setminus S)|}$

Two notions of phase transition

Definition I. Decay of long range correlations (informal)

Let e and f be any edges that are "far apart". Then in a random cut,

$$\Pr[\text{edge } e \text{ is cut} \mid \text{edge } f \text{ is cut}] \approx \Pr[\text{edge } e \text{ is cut}]$$

The study of algorithms based on correlation decay (notably, Weitz's algorithm) has been fruitful

Definition II. Analyticity of free energy (informal)

The "free energy" $\log Z$ is analytic in a complex neighborhood.

- Analyticity ≈ continuity of *observables*: the average cut size is precisely $\beta \cdot \frac{d \log Z}{d\beta}$
- Analyticity of free energy ≡ absence of zeros
- Even when only *positive real-valued parameters* make physical sense, complex-valued parameters are essential to the study of phase transitions
- Algorithmic use of location of zeros originated only recently in the work of Barvinok

Question

What relationship, if any, do the two notions (decay of correlations and zero-freeness) have? For example, can we use one to prove the other?

Prior works, and Lee-Yang vs Fisher zeros

Lee-Yang zeros: view λ as variable

$$Z_G^\beta(\lambda) = \sum_{S \subseteq V} \beta^{|E(S, V \setminus S)|} \lambda^{|S|}$$

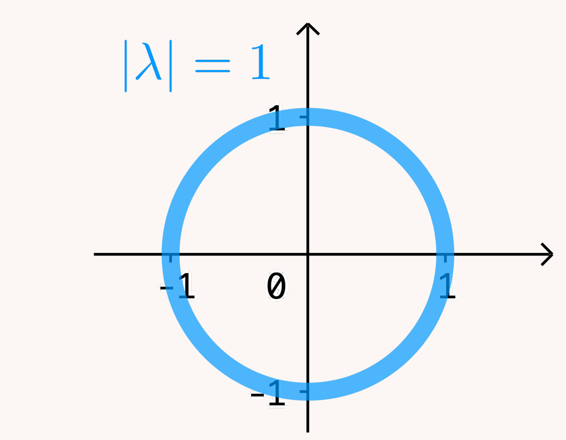

Fig. 3: Lee-Yang theorem: if $\beta < 1$, every zero is on the unit circle $|\lambda| = 1$

Fisher zeros: view β as variable

$$Z_G(\beta) = \sum_{S \subseteq V} \beta^{|E(S, V \setminus S)|}$$

Unlike LY, far from being well-understood

- For general Fisher zeros, Barvinok and Soberón: $Z_G(\beta) \neq 0$ if $|\beta - 1| < c/\Delta$, for $c \approx 0.34$
- Recently Peters and Regts: In the hard-core model, zero-free regions can be extended to the *entire* correlation decay regime

Our result: correlation decay implies zero-freeness in the Ising model

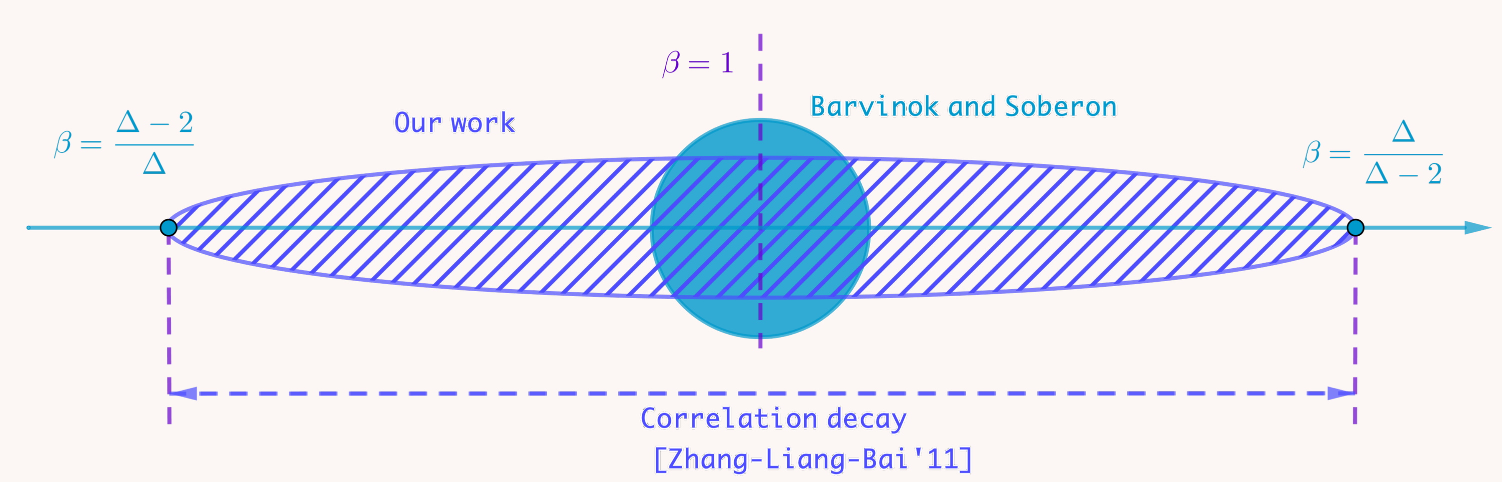

Fig. 4: Zero-free regions for Fisher zeros (illustrative only)

Theorem. correlation decay implies zero-freeness in the Ising model

$Z_G(\beta)$ does not vanish in a complex open region containing the entire correlation decay interval $B := \left(\frac{\Delta-2}{\Delta}, \frac{\Delta}{\Delta-2}\right)$.

By-product: efficient algorithms to approximate $Z_G(\beta)$ in the same region.

Remark. Our proof crucially exploits the correlation decay property!

Our technique: Weitz's algorithm

- Choose any vertex, say u, then

$$Z_G(\beta) = \sum_{S \subseteq V} \beta^{|E(S, V \setminus S)|} = \sum_{\substack{S \subseteq V \\ u \in S}} \beta^{|E(S, V \setminus S)|} + \sum_{\substack{S \subseteq V \\ u \notin S}} \beta^{|E(S, V \setminus S)|} = \Sigma_+ + \Sigma_-$$

- Consider the ratio

$$R_{G,u}(\beta) := \frac{\Sigma_+}{\Sigma_-}$$

- To show $Z_G(\beta) \neq 0$, it suffices that $\Sigma_- \neq 0$ and $R_{G,u}(\beta) \neq -1$

Weitz's algorithm provides a recurrence $F(\cdot)$ for computing the ratio $R_{G,u}(\beta)$

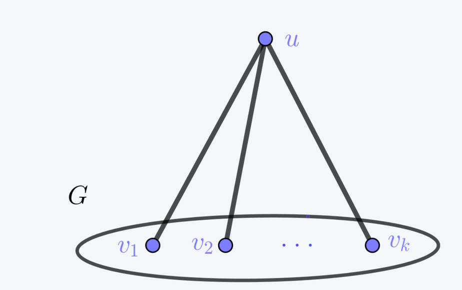

Fig. 5: The graph G and a vertex u

Given the ratios at $v_1, \cdots, v_k$, then the ratio at u is given by $R_{G,u} = F\left(R_{G_1,v_1}, \cdots, R_{G_k,v_k}\right)$, where

$$F_{\beta,k,s}(\vec{x}) := \beta^s \prod_{i=1}^{k} \frac{\beta + x_i}{\beta x_i + 1}$$

Proof sketch

To show $R_{G,u} \neq -1$, it suffices to design a complex neighborhood D such that

1. $F(D^k) \subseteq D$
2. $-1 \notin D$
3. D contains all the "starting points" of Weitz's algorithm

To find such a set D, the key steps are:

- For "convex" region D, the univariate recurrence $f(\cdot)$ satisfies $f(D) = F(D^k)$
- Therefore it suffices to show $f(D) \subseteq D$
- For a suitable choice of potential function φ, we show that $\varphi \circ f \circ \varphi^{-1}$ *approximately* contracts every rectangular region that contains the fixed point $\varphi(1)$.
- We choose a "convex" D so that $\varphi(D) \approx$ a rectangular region for every real valued $\beta \in B$, then we show that our proof is robust under *set approximation*
- As a result, for every $\beta \in B$, there is a constant sized complex neighborhood in which D_β still works for complex β' close enough to β

Open Problems

Is "*correlation decay implies absence of zeros*" a general phenomenon in spin systems and graphical models?

Connections of locations of zeros, to algorithms such as MCMC and the correlation decay approach?