Fisher zeros and correlation decay in the Ising model

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Motivation

In the past few years:

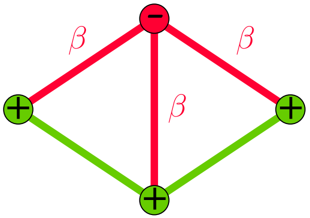
Phase transitions in statistical physics $\rightarrow$ algorithms

In this work, we study the converse:

Can we study phase transitions in statistical physics via algorithmic techniques?

Ising model

- Configuration: $\sigma \in \{+,-\}^V$
- Edge potentials: $\varphi_e(\sigma_u,\sigma_v) = \begin{cases} \beta & \text{if } \sigma_u \neq \sigma_v \\ 1 & \text{otherwise} \end{cases}$



A spin configuration with weight β^3

Ising model as cut generating polynomial

$$Z_G(\beta) = \sum_{S\subseteq V} \beta^{|E(S,V\setminus S)|} = \sum_{k=0}^{|E|} \gamma_k \beta^k$$

where γ_k := number of k-edge cuts

- Gibbs distribution: $\Pr[(S, V \setminus S)] = \frac{1}{Z_G(\beta)} \cdot \beta^{|E(S,V\setminus S)|}$

Two notions of phase transition in statistical physics

Definition I. Decay of long range correlations (informal)

Let e and f be any edges that are "far apart". Then in a random cut,

$$\Pr[\text{edge } e \text{ is cut} \mid \text{edge } f \text{ is cut}] \approx \Pr[\text{edge } e \text{ is cut}]$$

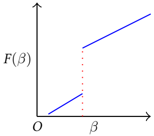
The study of algorithms based on correlation decay (notably, Weitz's algorithm) has been fruitful

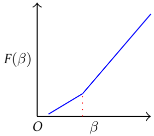
Two notions of phase transition in statistical physics

Definition II. Analyticity of free energy (informal)

The "free energy" $\log Z$ is analytic in a complex neighborhood.

- Analyticity $\approx$ continuity of *observables*: the average cut size is precisely $\beta \cdot \frac{d \log Z}{d\beta}$





- Analyticity of $\log Z \equiv$ absence of zeros in Z
- Even when only *positive real-valued parameters* make physical sense, one needs to study complex-valued parameters
- Algorithmic use of location of zeros originated only recently in the work of Barvinok

What relationship, if any, do the two notions (decay of correlations and zero-freeness) have?

Prior works, and Lee-Yang zeros versus Fisher zeros

Fisher zeros has been studied classically, but little is known for general graphs

Fisher zeros (1965): view β as variable

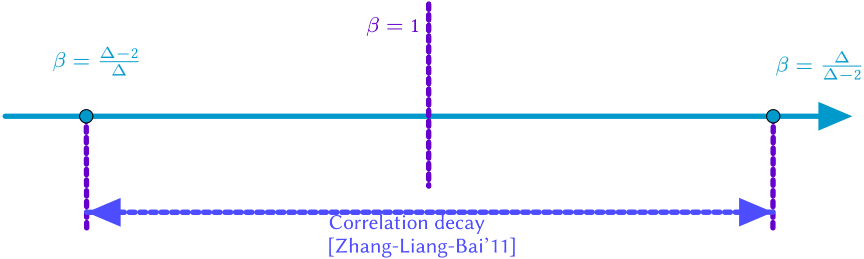
$$Z_G(\beta) = \sum_{S \subseteq V} \beta^{|E(S, V \setminus S)|}$$

Lee-Yang zeros (1952): view λ as variable

$$Z_G^{\beta}(\lambda) = \sum_{S \subseteq V} \beta^{|E(S, V \setminus S)|} \lambda^{|S|}$$

- For general Fisher zeros, Barvinok and Soberón: $Z_G(\beta) \neq 0$ if $|\beta - 1| < c/\Delta$, for $c \approx 0.34$
- Recently Peters and Regts: in the hard-core model, zero-free regions can be extended to the *entire* correlation decay regime

Our result: correlation decay implies zero-freeness for the Ising model



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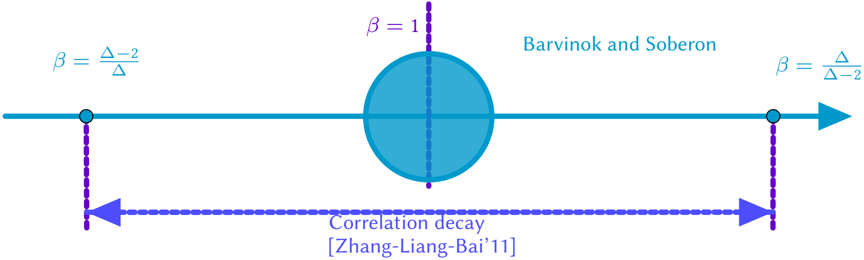

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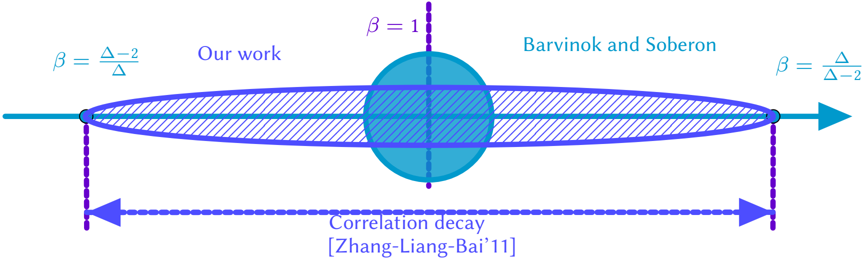


Theorem

$Z_G(\beta)$ does not vanish in a complex open region containing the entire correlation decay interval $B := \left(\frac{\Delta-2}{\Delta}, \frac{\Delta}{\Delta-2}\right)$.

By-product: algorithms to approximate $Z_G(\beta)$ in the same region.

Our technique: Weitz's algorithm

Our proof crucially exploits the correlation decay property

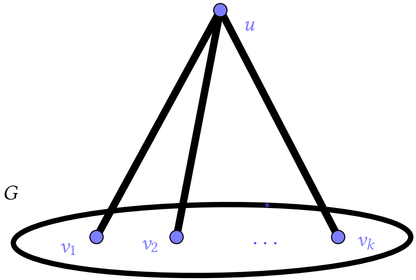
- Choose any vertex, say u, then

$$Z_G(\beta) = \sum_{S \subseteq V} \beta^{|E(S,V\setminus S)|} = \sum_{\substack{S \subseteq V \\ u \in S}} \beta^{|E(S,V\setminus S)|} + \sum_{\substack{S \subseteq V \\ u \notin S}} \beta^{|E(S,V\setminus S)|} = \Sigma_+ + \Sigma_-$$

- Consider the ratio $R_{G,u}(\beta) := \frac{\Sigma_+}{\Sigma_-}$.
- To show $Z_G(\beta) \neq 0$, it suffices if $\Sigma_- \neq 0$ and $R_{G,u}(\beta) \neq -1$

Weitz's algorithm provides a formal recurrence $F(\cdot)$ for computing the ratio $R_{G,u}(\beta)$

Our technique (Weitz's algorithm cont'd)



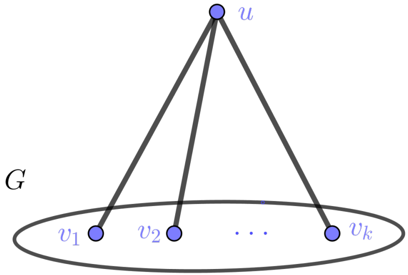
Given the ratios at $v_1, \cdots, v_k$, then the ratio at u is given by $R_{G,u} = F(R_{G_1,v_1}, \cdots, R_{G_k,v_k})$, where

$$F_{\beta,k,s}(\vec{x}) := \beta^s \prod_{i=1}^{k} \frac{\beta + x_i}{\beta x_i + 1}$$

Proof sketch

To show $R_{G,u} \neq -1$, it suffices to design a complex neighborhood D such that

1. $F(D^k) \subseteq D$
2. $-1 \notin D$
3. D contains all the "starting points" of Weitz's algorithm



$$R_{G,u} = F(R_{G_1,v_1}, \cdots, R_{G_k,v_k})$$

To find such a set, the key steps are:

- For "convex" region D, the univariate recurrence $f(\cdot)$ satisfies $f(D) = F(D^k)$
- For a suitable choice of φ, we show that $\varphi \circ f \circ \varphi^{-1}$ *approximately* contracts every rectanglular region that contains the fixed point $\varphi(1) \longleftarrow$ **Correlation decay!**
- We choose a "convex" D so that $\varphi(D) \approx$ a rectangular region

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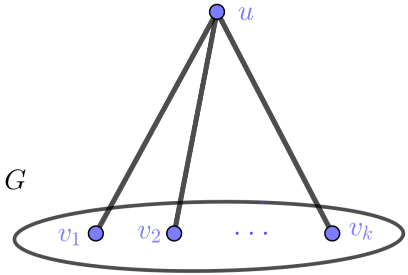

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Discussion and Open problems

Open problem

Is "*correlation decay implies absence of zeros*" a general phenomenon in spin systems and graphical models?

Open problem

Connections of locations of zeros, to algorithms such as MCMC and the correlation decay approach?