Zeros of ferromagnetic 2-spin systems

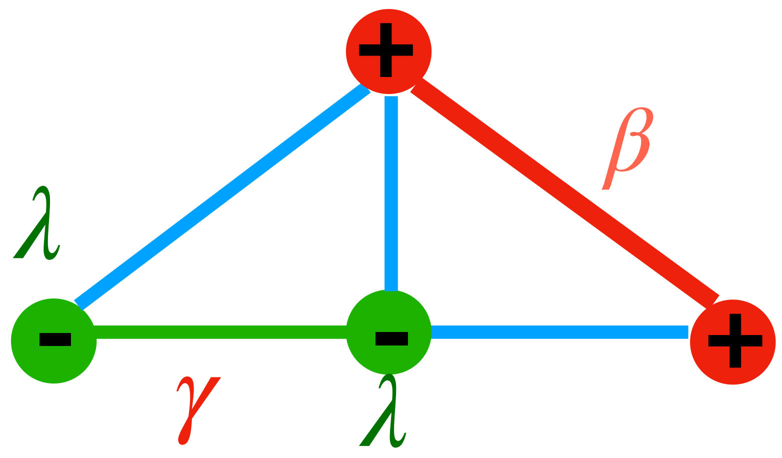
Jingcheng Liu (Caltech)

Joint work with **Heng Guo** (University of Edinburgh) and
Pinyan Lu (Shanghai University of Finance and Economics)

Spin systems

Given a graph $G = (V, E)$

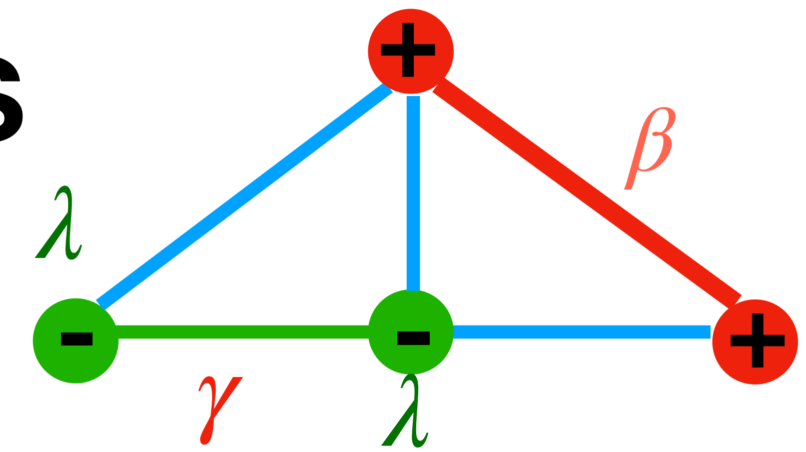
- Configuration $\sigma : V \to \{+, -\}$
- Edge interactions
  - '+ +' edge: β
  - '- -' edge: γ
  - '+ -' edge: 1



- External field λ for every vertex assigned '-'
- The partition function

$$Z_G(\beta, \gamma, \lambda) := \sum_{\sigma : V \to \{+,-\}} \beta^{(\text{'+ +' edges})} \gamma^{(\text{'- -' edges})} \lambda^{(\text{\# '-' -spin vertices})}.$$

Spin systems



Given a graph $G = (V, E)$

$$Z_G(\beta, \gamma, \lambda) := \sum_{\sigma : V \to \{+,-\}} \beta^{(\text{'+ +' edges})} \gamma^{(\text{'- -' edges})} \lambda^{(\text{\# '-' -spin vertices})}.$$

- Gibbs distribution

$$\Pr[\sigma] = \frac{1}{Z_G(\beta, \gamma, \lambda)} \cdot \beta^{(\text{'+ +' edges})} \gamma^{(\text{'- -' edges})} \lambda^{(\text{\# '-' -spin vertices})}$$

- Ferromagnetic if $\beta\gamma > 1$: favors agreements
- WLOG, assume $\beta \geq \gamma$
- $\beta = \gamma$: Ising model

Approximate counting

- Compute $(1 \pm \varepsilon) \cdot Z_G$
- Equivalent to
  - **(Approximate) sampling**

    Sampling from the Gibbs distribution?

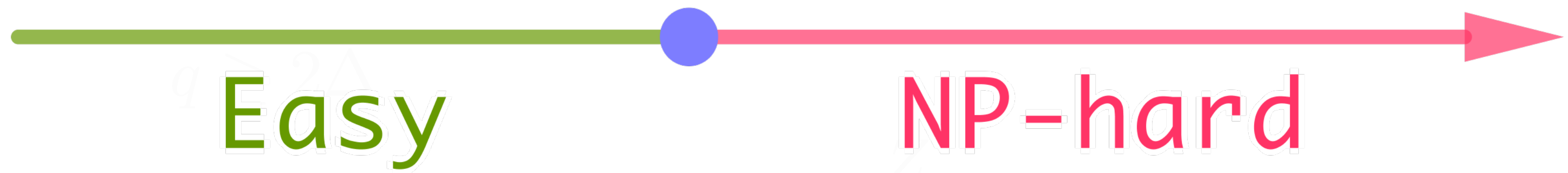
  - **Approximate inference**

    Given partial observation of the system, what can you infer about the rest?

  - **Approximate root-finding**
  - …

Prior work

- Antiferromagnetic regime: there is a threshold $\lambda_c(\beta, \gamma)$



[Sly-Sun], [Li-Lu-Yin], [Sinclair-Srivastava-Thurley]

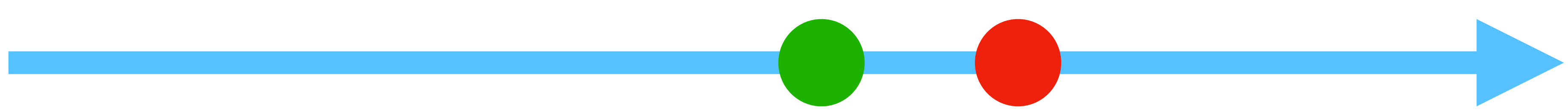
- Ising model: MCMC, Barvinok's interpolation

[Jerrum-Sinclair], [L.-Sinclair-Srivastava]

- Ferromagnetic 2-spin:

MCMC $\lambda \leq \frac{\beta}{\gamma}$ $\lambda \geq \left(\frac{\beta}{\gamma}\right)^{d'}$ **#BIS-hard [L.-Lu-Zhang]**

Further, assuming $\gamma \leq 1$:



SSM [Guo-Lu] $\lambda \leq \left(\frac{\beta}{\gamma}\right)^{d_c}$ $\lambda \geq \left(\frac{\beta}{\gamma}\right)^{d'}$

Our main result

Main algorithmic result:

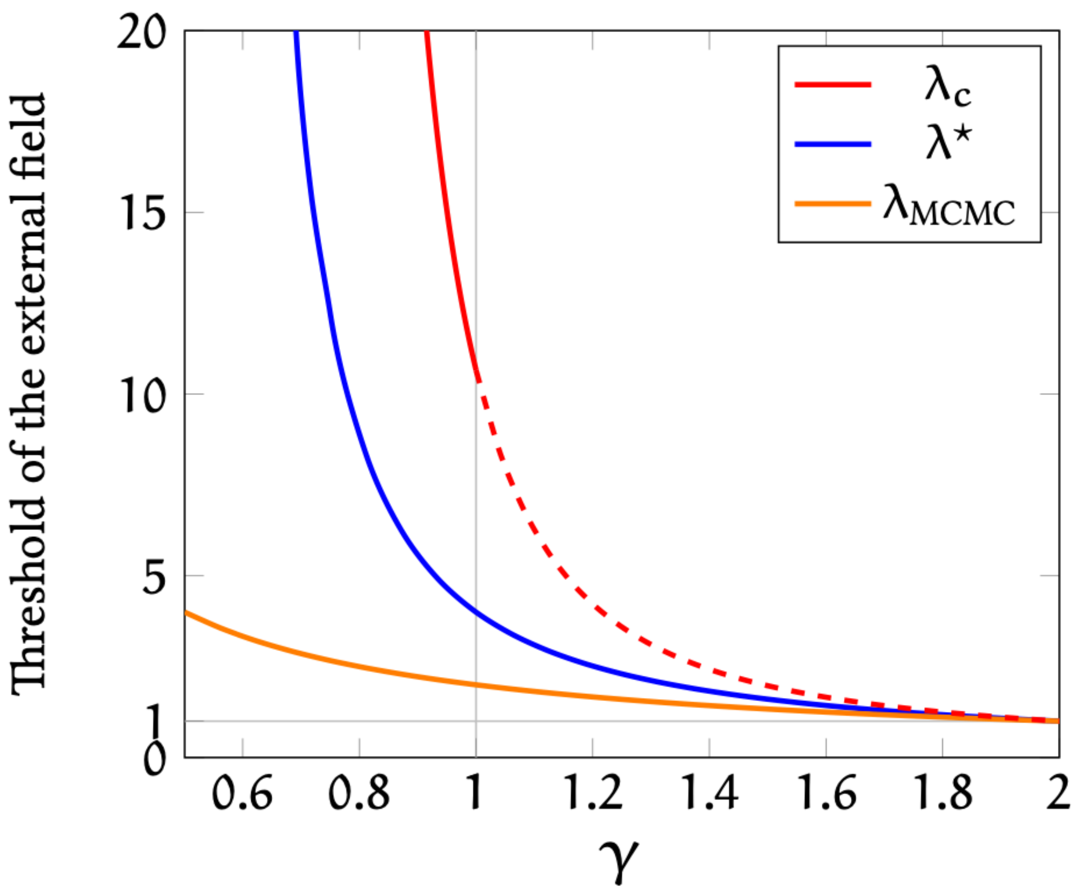
Fix any $\beta, \gamma > 0$ and $\beta\gamma \geq 1$, $\beta \geq \gamma$. If $\lambda < \left(\frac{\beta}{\gamma}\right)^{d^\star/2}$, then there exists a deterministic ***FPTAS***, which outputs $\hat{Z}$ s.t.

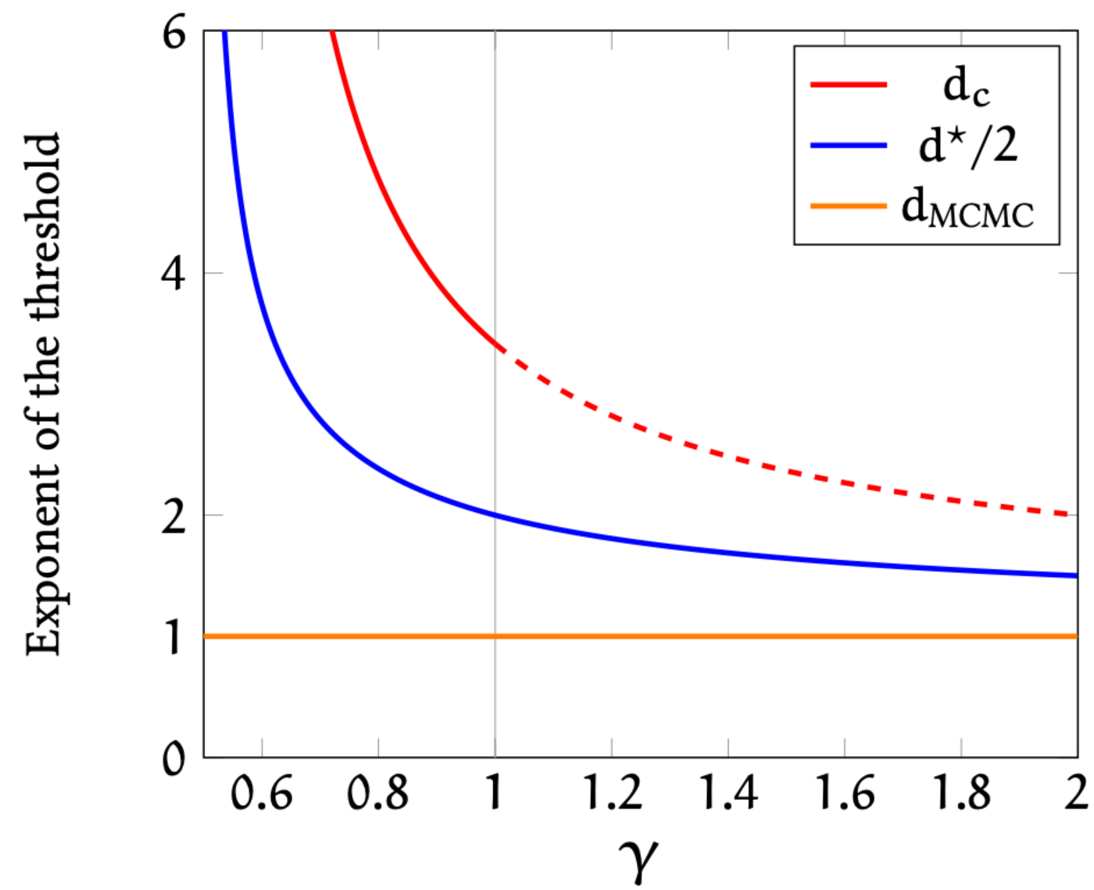
$$\hat{Z} \in (1 \pm \varepsilon) \cdot Z_G$$

in time $\mathrm{poly}(|G|, 1/\epsilon)$ for any bounded degree graph G

$$\lambda^\star := \left(\frac{\beta}{\gamma}\right)^{d^\star/2}$$

$$d^\star := \frac{\pi}{\tan^{-1}\sqrt{\beta\gamma - 1}}$$





Barvinok's interpolation

$$Z_G(\beta, \gamma, \lambda) := \sum_{\sigma: V \to \{+,-\}} \beta^{(\text{'+ +' edges})} \gamma^{(\text{'- -' edges})} \lambda^{(\text{\# '-' -spin vertices})}.$$

Fix β, γ, view Z_G as a polynomial in λ

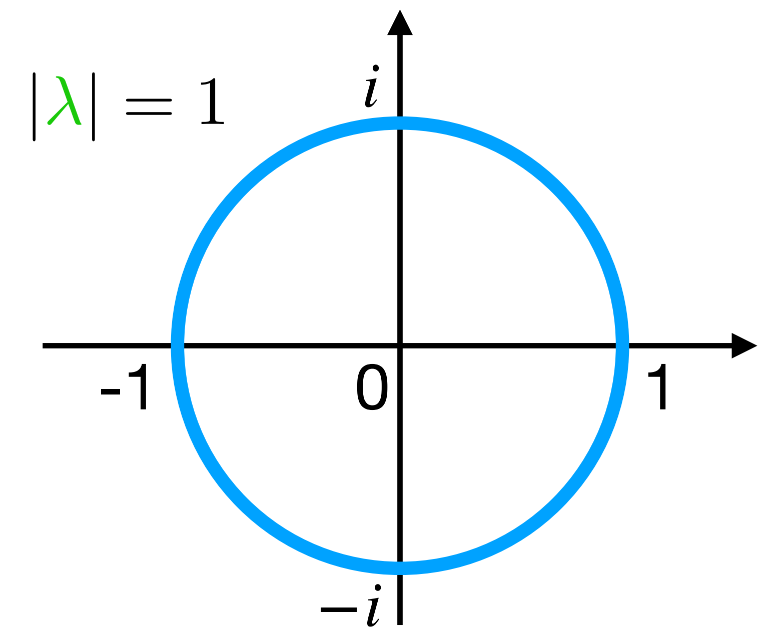
Key: leverage that Z_G does not vanish in certain complex region

[Barvinok, Barvinok and Soberon]

- **Consider the Taylor expansion of $\log Z_G$**
- **In a zero free region, $\log Z_G$ can be approximated to $\pm\epsilon$ by its k-th order Taylor series for $k = O(\log(n/\epsilon))$**
- $\log Z_G \pm \epsilon \iff (1 \pm \epsilon) \cdot Z_G$
- **k-th order Taylor series is determined by the first k+1 coefficients of Z**
- **Naively computing the first k+1 coefficients of Z takes time $O(n^k)$**
- $\implies$ **Quasi-polynomial time algorithm for $k = O(\log(n/\epsilon))$**
- **Exploiting the combinatorial structure speeds up to $O(n(e\Delta)^k)$**

Lee-Yang zeros of Ferromagnetic 2-spin systems

$$Z_G(\beta,\gamma,\lambda) := \sum_{\sigma:V\to\{+,-\}} \beta^{(\text{'+ +' edges})}\gamma^{(\text{'- -' edges})}\lambda^{(\text{\# '-' -spin vertices})}.$$



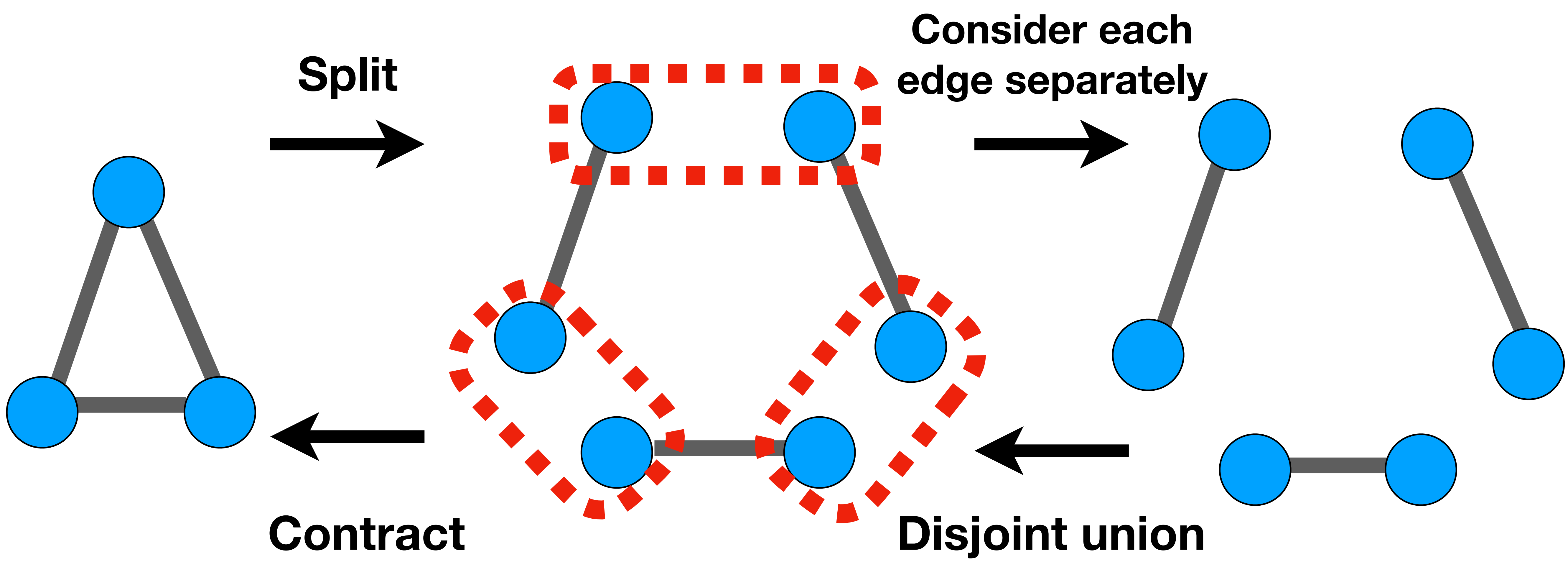
Lee-Yang theorem for the Ising model

Let $\beta \geq 1$, then $Z_G(\beta,\beta,\lambda) = 0 \implies |\lambda| = 1$

Main technical result:

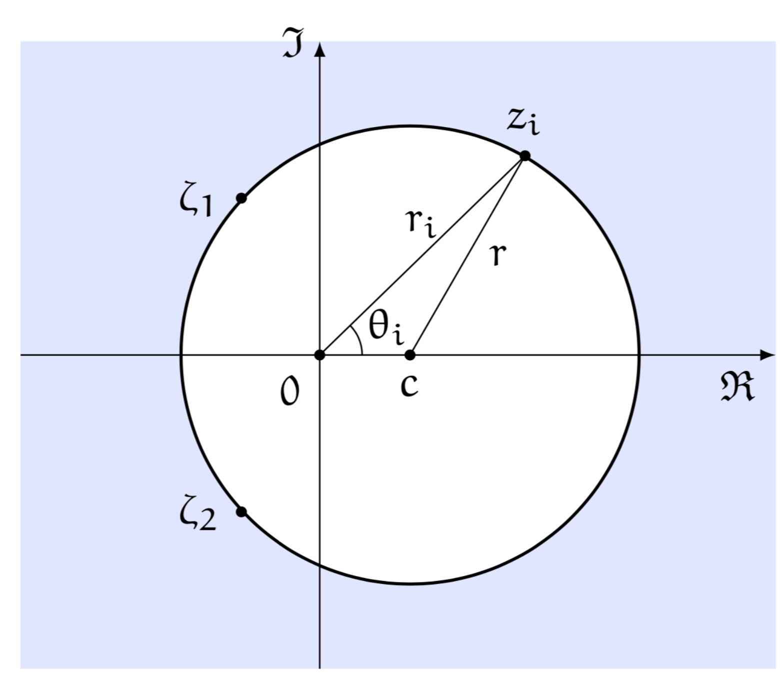
Fix any $\beta, \gamma > 0$ and $\beta\gamma > 1$, $\beta \geq \gamma$. For any graph G with minimum degree 2, $Z_G(\beta,\gamma,\lambda)$, viewed as a polynomial in λ, does not vanish in a constant sized neighborhood containing $\left[0, \left(\frac{\beta}{\gamma}\right)^{d^\star/2}\right)$

Asano’s contraction method



Locate complex zeros after contraction

Zero-free regions are preserved under disjoint union



Multivariate partition function

$$Z_G(\beta,\gamma,\lambda) := \sum_{\sigma:V\to\{+,-\}} \beta^{(\text{`+ +' edges})}\gamma^{(\text{`- -' edges})}\lambda^{(\text{\# `-' -spin vertices})}.$$

Multivariate: $$Z_G(\beta,\gamma,\vec{\lambda}) := \sum_{S\subseteq V} \beta^{|E[S^c]|}\gamma^{|E[S]|}\prod_{v\in S}\lambda_v.$$

Asano's contraction: base case

$$Z_G(\beta, \gamma, \vec{\lambda}) := \sum_{S \subseteq V} \beta^{|E[S^c]|} \gamma^{|E[S]|} \prod_{v \in S} \lambda_v.$$

The partition function on a single edge $\gamma\lambda_1\lambda_2 + \lambda_1 + \lambda_2 + \beta$

Consequence of the Grace–Walsh–Szegő coincidence theorem

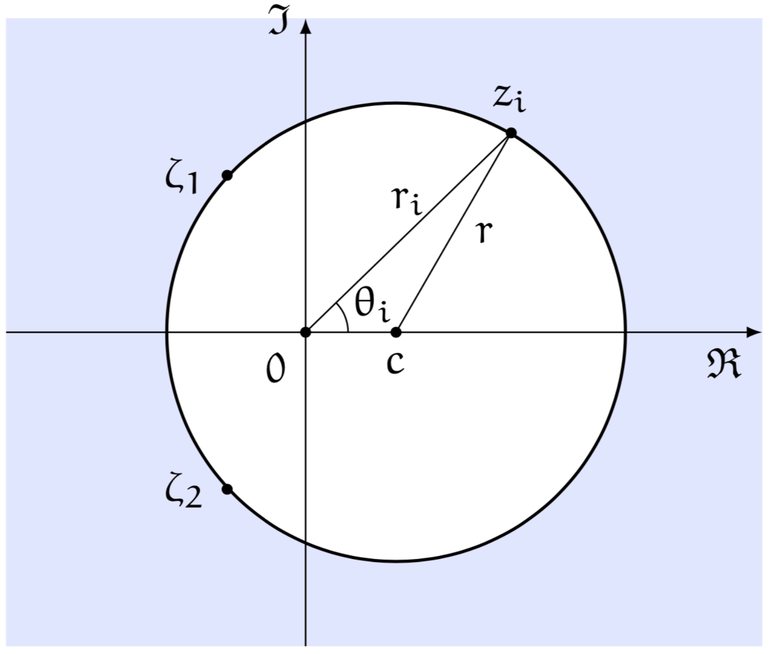
Let ζ_1, ζ_2 be the two complex roots of

$$\gamma\lambda^2 + 2\lambda + \beta = 0.$$

Then for any closed circular region K containing ζ_1, ζ_2, the polynomial

$$\gamma\lambda_1\lambda_2 + \lambda_1 + \lambda_2 + \beta$$

can only vanish if either $\lambda_1 \in K$ or $\lambda_2 \in K$



Asano's contraction: invariants

$$Z_G(\beta, \gamma, \vec{\lambda}) := \sum_{S \subseteq V} \beta^{|E[S^c]|} \gamma^{|E[S]|} \prod_{v \in S} \lambda_v.$$

Minkowski product of sets

For sets $A, B \subseteq \mathbb{C}$,

$$A \cdot B := \{a \cdot b : a \in A, b \in B\}$$

$$A^d := \left\{ \prod_{i=1}^{d} a_i : \forall i, a_i \in A \right\}$$

For carefully chosen regions $K(\beta, \gamma)$, we show the following is preserved under contraction

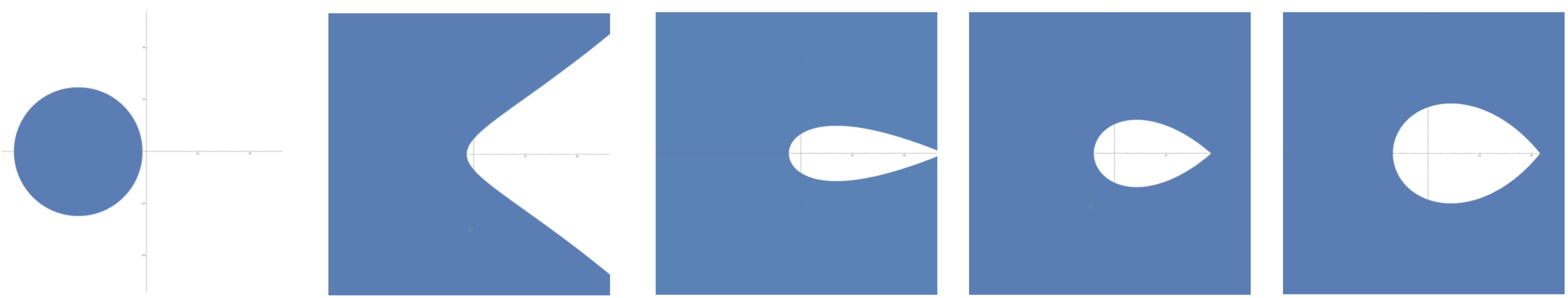
Main technical lemma:

Fix any $\beta, \gamma > 0$ and $\beta\gamma > 1$, $\beta \geq \gamma$. The partition function $Z_G(\beta, \gamma, \vec{\lambda})$ can vanish only if $\exists i : \lambda_i \in (-1)^{d+1} \cdot K^d$, where $d = \deg_G(i)$

Minkowski product of circular regions

<u>Considerations in choosing the region $K(\beta, \gamma)$:</u>

- $K(\beta, \gamma)$ needs to contain ζ_1 and ζ_2 (the two complex roots in the base case)
- maximize the intersection of zero-free regions



$\mathsf{K} = \overline{\mathcal{D}(c, r)}$ $\quad$ $\mathcal{K}_2 = -\mathsf{K} \cdot \mathsf{K}$ $\quad$ $\mathcal{K}_3 = \mathsf{K}^3$ $\quad$ $\mathcal{K}_4 = -\mathsf{K}^4$ $\quad$ $\mathcal{K}_5 = \mathsf{K}^5$

FIGURE 4. Our region $\mathsf{K} = \overline{\mathcal{D}(c, r)}$, $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$ and $\mathcal{K}_5$ in the case of $\beta = 4$ and $\gamma = \frac{1}{2}$. Here, the intercept of $\mathcal{K}_d$ on the positive real line is minimised at $d = 4$ for all $d \geqslant 2$.

Future directions

- Proving zero-freeness all the way up to λ_c
  - The current threshold is the best one can get with the invariant that we chose
- What is the correct threshold?
  - The gadget construction in the hardness proof introduces an integrality gap
  - λ_c is not correct either: algorithms can be found beyond λ_c

Thanks

Q & A